

Generation of heterotic string theory solutions from the stationary Einstein-Maxwell fields

Oleg V. Kechkin

DEPNI, Institute of Nuclear Physics
M.V. Lomonosov Moscow State University
119899 Moscow, Vorob'yovy Gory, Russia
e-mail: kechkin@depni.npi.msu.su

Abstract

A new formalism for construction of the stationary solutions is developed for the four-dimensional gravity coupled to the dilaton, Kalb-Ramond and two Maxwell fields in a low-energy heterotic string theory form. The result of generation is automatically invariant in respect to subgroup of the stationary charging symmetry transformations; the generation can be started from the stationary Einstein-Maxwell fields. The formalism is given both in real and new compact complex form, the result of maximal symmetry extension of the stationary Einstein-Maxwell theory to discussing string gravity model is explicitly written down.

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1 Introduction

String theory leads to field theories of gravity coupled to different matter fields in its low energy limit [1]. For the heterotic string theory case these matter fields include a set of Abelian vector ones, so the corresponding gravity model can be considered as some string theory generalization of the classical Einstein-Maxwell theory [2].

The heterotic string theory motivated gravity model possesses a rich system of on-shell symmetries [3], [4], [5], [6], [7], [8]. These symmetries can be used for generation of new solutions from the known ones accordingly the classical Sofus Lee approach. To apply the symmetry transformation technique one must find some special truncation of the studying theory which in fact is the system with well known solution space. In this case it begins possible to construct the symmetry extension of this subsystem to the whole theory. Such a program had been extensively realized in the heterotic string theory framework for the choice of stationary General Relativity as the starting subsystem [9], [10].

In this article the stationary Einstein-Maxwell theory will put down to work. This step is nontrivial; to clarify the question let us consider the stringy dilaton-axion generalization of Einstein-Maxwell theory. The corresponding action includes the terms $e^{\alpha\Phi}F^2$ and $\beta\kappa F\tilde{F}$, where F is the Maxwell stress tensor, Φ and κ are the dilaton and Pecci-Quinn axion fields, whereas α and β are the dilaton-Maxwell and axion-Maxwell constant couplings. For the realistic string theory motivated gravity models $\alpha, \beta \neq 0$, so if one puts $\Phi = \kappa = 0$ on shell to obtain the Einstein-Maxwell theory from the discussing model, one additionally obtains the restrictions

$$F^2 = F\tilde{F} = 0 \tag{1.1}$$

from the original ‘ Φ ’ and ‘ κ ’-equations. These restrictions leave only the plane-wave special case for the possibility of string theory generalization of the Einstein-Maxwell theory which is not sufficient for the practice. However, as it will be shown below, it exists another form of enclosure of Einstein-Maxwell theory into the heterotic string gravity models which is free of any restrictions. This new approach is directly related to some new formalism based on the use of matrix Ernst-type potentials which had been developed for the general heterotic string gravity models in [8], [11]. Below it will be explored for the total extension of the stationary Einstein-Maxwell theory to the heterotic string theory case in respect to its subgroup of the stationary charging symmetry transformations. The developed formalism provides a compact and convenient tool for the generation of charged and asymptotically flat solutions in the low-energy heterotic string theory.

2 New matrix formalism and charging symmetries

Let us now formulate the heterotic string gravity model under consideration. The starting effective field theory lives in the $(d+3)$ -dimensional space-time with the signature $+\cdots+; -, +, +$ and describes the bosonic zero-mass modes of excitation of the heterotic string theory: the dilaton Φ , Kalb-Ramond field B_{MN} ($M, N = 1, \dots, d+3$), n Abelian fields A_M^I ($I = 1, \dots, n$) and the metric G_{MN} . The corresponding action is [1]:

$$S_{d+3} = \int dX \sqrt{-\det G_{MN}} e^{-\Phi} \left(R_{d+3} + \Phi_{,M} \Phi^{,M} - \frac{1}{12} H_{MKN} H^{MKN} - \frac{1}{4} F_{MN}^I F^{IMN} \right), \quad (2.1)$$

where $H_{MKN} = B_{NK,M} - 1/2 F_{NK}^I A_M^I + \text{cyclic}(M, N, K)$ and $F_{MN}^I = A_{N,M}^I - A_{M,N}^I$. Following [3], [4],[5] we consider a toroidal compactification of the first d dimensions. In this case all the fields begin independent on the coordinates X^m with $m = 1, \dots, d$ and depend on the ones $x^\mu = X^{d+\mu}$, $\mu = 1, 2, 3$. The resulting 3-dimensional system is equivalent on-shell to some concrete nonlinear σ -model coupled to gravity [5]. Such σ -models which parametrize a coset space had been classified in [12]; for the theory under consideration the coset is $O(d+1, d+1+n)/O(d+1) \times O(d+1+n)$ [5]. The resulting 3-dimensional gravity model describes the set of $(d+1) \times (d+1+n)$ functionally independent scalar fields coupled to the effective 3-metric $h_{\mu\nu}$. It can be alternatively described in terms of the null-curvature $[2(d+1)+n] \times [2(d+1)+n]$ matrix \mathcal{M} [5],[7], the pair of matrix Ernst potentials \mathcal{X} and \mathcal{A} of the dimension $(d+1) \times (d+1)$ and $(d+1) \times n$ [7], and also using the $(d+1) \times (d+1+n)$ -dimensional matrix potential \mathcal{Z} [8], [11]. In Appendix A one can find the definition of σ -model scalar fields and the relations between these three alternative matrix formulations.

For our purposes we need in the \mathcal{Z} -formulation; the effective 3-dimensional action reads [11]:

$$S_3 = \int dx \left\{ -R_3 + \text{Tr} \left[\nabla \mathcal{Z} \left(\Xi - \mathcal{Z}^T \Sigma \mathcal{Z} \right)^{-1} \nabla \mathcal{Z}^T \left(\Sigma - \mathcal{Z} \Xi \mathcal{Z}^T \right)^{-1} \right] \right\}, \quad (2.2)$$

where Σ and Ξ are $(d+1) \times (d+1)$ and $(d+1+n) \times (d+1+n)$ matrices of the form $\text{diag}(-1, -1; 1, \dots, 1)$. In [11] one can find full information about this representation; here we need only in trivial fact that the transformation

$$\mathcal{Z} \rightarrow \mathcal{C}_L \mathcal{Z} \mathcal{C}_R \quad (2.3)$$

is a symmetry of the action (2.2) if

$$\mathcal{C}_L^T \Sigma \mathcal{C}_L = \Sigma, \quad \mathcal{C}_R^T \Xi \mathcal{C}_R = \Xi, \quad (2.4)$$

i.e. if $\mathcal{C}_L \in O(2, d-1)$ and $\mathcal{C}_R \in O(2, d-1+n)$. In [8] it is shown that this symmetry is equivalent to the charging symmetry subgroup of the complete group of 3-dimensional symmetries of the theory. The transformations from the charging symmetry subgroup leave unchanged the trivial solution $\mathcal{Z} = 0$, $h_{\mu\nu} = \delta_{\mu\nu}$ of motion equations corresponding the action (2.2). This trivial 3-dimensional solution, being expressed in terms of the original multidimensional string theory fields, arises in framework of the asymptotically flat solutions or solutions possessing the nontrivial NUT, magnetic and other similar charges (see [2] for the Einstein-Maxwell analogies). The asymptotically flat solutions play extremely important role in physical interpretations of the theory and often (for example, in the black hole physics [9], [13]) the charging symmetry subgroup is the only resultative part of the complete group of symmetries. The remaining part of symmetries provide the nonimportant shift of values of the physical fields at the spatial infinity.

3 Generation using real potentials

Now let us consider the special case of $d = 1$, $n = 2$, which is also interesting in framework of the $D = N = 4$ supergravity [14]. Also let us consider the charging symmetry transformations which can be continuously related to the identical one, i.e. let $\mathcal{C}_L \in SO(2)$, $\mathcal{C}_R \in SO(2, 2)$ in the following consideration. The nearest goal is to write down these transformation matrices in the explicit form, which can be obtained through exponentiation of the corresponding infinitesimal transformations.

So, let $\hat{\lambda}_L$ and $\hat{\lambda}_R$ are the general $so(2)$ and $so(2, 2)$ algebra matrices, i.e. for the group matrices one has

$$\mathcal{C}_L = \exp \hat{\lambda}_L, \quad \mathcal{C}_R = \exp \hat{\lambda}_R. \quad (3.1)$$

The algebraic matrices can be calculated as the general linear combinations of the corresponding generators,

$$\hat{\lambda}_L = \lambda_0 \Gamma_0, \quad \hat{\lambda}_R = \lambda_K \Gamma_K, \quad (3.2)$$

where the generators satisfy the algebraic relations

$$\Gamma_0^T = -\Gamma_0, \quad \Gamma_K^T = -\Sigma_3 \Gamma_K \Sigma_3, \quad (3.3)$$

with

$$\Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.4)$$

The generator Γ_0 is the 2×2 matrix, whereas Γ_K are the 4×4 ones; they can be taken in the following form:

$$\begin{aligned}\Gamma_0 &= \epsilon = -i\sigma_2; \\ \Gamma_1^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2^1 = \begin{pmatrix} 0 & -\epsilon \\ \epsilon & 0 \end{pmatrix}, \quad \Gamma_3^1 = \begin{pmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}; \\ \Gamma_1^2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \quad \Gamma_2^2 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \Gamma_3^2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix},\end{aligned}\tag{3.5}$$

where the usual Pauli matrices have been used (in our notations $\{\Gamma_K\} = \{\Gamma_\mu^1, \Gamma_\nu^2\}$). Then, it is easy to verify that

$$[\Gamma_\mu^1, \Gamma_\nu^2] = 0\tag{3.6}$$

and that the following multiplication relations take place:

$$\begin{aligned}(\Gamma_1^a)^2 &= (\Gamma_2^a)^2 = -(\Gamma_3^a)^2 = 1, \\ \Gamma_1^a \Gamma_2^a &= -\Gamma_2^a \Gamma_1^a = \Gamma_3^a, \\ \Gamma_2^a \Gamma_3^a &= -\Gamma_3^a \Gamma_2^a = -\Gamma_1^a, \\ \Gamma_3^a \Gamma_1^a &= -\Gamma_1^a \Gamma_3^a = -\Gamma_2^a,\end{aligned}\tag{3.7}$$

where $a = 1, 2$. Thus, $\hat{\lambda}_R = \hat{\lambda}_R^1 + \hat{\lambda}_R^2$, where $\hat{\lambda}_R^a = \lambda_\mu^a \Gamma_\mu^a$, and from Eqs. (3.6), (3.7) it follows that the matrices $\hat{\lambda}^a$ parametrize two commuting 4×4 realizations of the $so(2, 1)$ algebra. This fact is related to the isomorphism $so(2, 2) \sim so(2, 1) \oplus so(2, 1)$ and will be crucial for our analysis. Using Eqs. (3.6) and (3.7) it is possible to calculate \mathcal{C}_R ; the result reads:

$$\mathcal{C}_R = \mathcal{C}_R^1 \mathcal{C}_R^2 = \mathcal{C}_R^2 \mathcal{C}_R^1,\tag{3.8}$$

where

$$\mathcal{C}_R^a = \cosh \lambda^a + \frac{\sinh \lambda^a}{\lambda^a} \tilde{\lambda}^a\tag{3.9}$$

and $\lambda^a = \{(\lambda_1^a)^2 + (\lambda_2^a)^2 - (\lambda_3^a)^2\}^{1/2}$. For the matrix \mathcal{C}_L one has:

$$\mathcal{C}_L = \cos \lambda_0 + \sin \lambda_0 \Gamma_0.\tag{3.10}$$

Finally, Eqs. (2.3), (3.8) (3.9) and (3.10) together with the definition of the potential \mathcal{Z} , given in Appendix A, completely define the generation using continuous charging symmetries in this theory in the explicit form.

Now let us define the starting subsystem for the generation procedure. To do it, let us put $\mathcal{Z} = (\mathcal{Z}_1, \mathcal{Z}_2)$, where \mathcal{Z}_a are the 2×2 block components. Let us also put

$$\mathcal{Z}_a = z'_a - z''_a \epsilon, \quad (3.11)$$

where

$$z_a = z'_a + i z''_a \quad (3.12)$$

is the complex function. The first statement crucial for the following consideration is that the pair $(\mathcal{Z}, h_{\mu\nu})$ gives the solution of motion equations for the theory (2.2) if the pair $(\mathbf{z}, h_{\mu\nu})$ is the solution for the theory

$$S_{EM} = \int dx \left\{ -R_3 + 2 \frac{\nabla \mathbf{z} (\sigma_3 - \mathbf{z}^+ \mathbf{z})^{-1} \nabla \mathbf{z}^+}{1 - \mathbf{z} \sigma_3 \mathbf{z}^+} \right\}, \quad (3.13)$$

where

$$\mathbf{z} = (z_1, z_2) \quad (3.14)$$

and the relation (3.11) takes place. The proof is related to the fact that the matrix (3.11) is the so called ‘exact’ realization of the complex function (3.12) and the multiplier ‘2’ is defined by the matrix dimensionality of this realization. Then, the absence of any additional restrictions follows from the closing character of complex number basis $(1, i)$. The second statement is that the theory (3.13) coincides with the stationary Einstein-Maxwell theory. To prove it, let us introduce the alternative complex variables

$$\mathcal{E} = \frac{1 - z_1}{1 + z_1}, \quad \mathcal{F} = \frac{\sqrt{2} z_2}{1 + z_1} \quad (3.15)$$

(it is interesting to note that the inverse map $(\mathcal{E}, \mathcal{F}) \rightarrow (z_1, z_2)$ has the same form as Eq. (3.15)). Then, in terms of these new potentials,

$$S_{EM} = \int dx \left\{ -R_3 + \frac{1}{2F^2} \left| \nabla \mathcal{E} + \bar{\mathcal{F}} \nabla \mathcal{F} \right|^2 - \frac{1}{f} |\mathcal{F}|^2 \right\}, \quad (3.16)$$

where $f = 1/2(\mathcal{E} + \bar{\mathcal{E}} + |\mathcal{F}|^2)$, i.e. exactly the stationary Einstein-Maxwell theory action [2], [15] with \mathcal{E} and \mathcal{F} as Ernst potentials [16].

The third important statement can be formulated as the nonimportance of the charging symmetry subgroups given by the matrices \mathcal{C}_L and \mathcal{C}_R^1 for generation starting from the

stationary Einstein-Maxwell theory. Actually, let us suppose that the solution $(\mathbf{z}, h_{\mu\nu})$ of the theory (3.13) is charging symmetry complete, i.e. the symmetry transformations preserving the trivial \mathbf{z} -value had been applied in framework of the Einstein-Maxwell theory itself. These transformations are obviously given by the single map

$$\mathbf{z} \rightarrow \mathbf{C}_L \mathbf{z} \mathbf{C}_R^1 \quad (3.17)$$

with $\mathbf{C}_L \in U(1)$ and $\mathbf{C}_R^1 \in SU(1, 1)$. It is easy to see that the substitution (matrices \rightarrow numbers)

$$1 \rightarrow 1, \quad -\epsilon \rightarrow i \quad (3.18)$$

generates the map $\Gamma_0 \rightarrow \mathbf{\Gamma}_0$, $\Gamma_\mu^1 \rightarrow \mathbf{\Gamma}_\mu$, where

$$\begin{aligned} \mathbf{\Gamma}_0 &= i, \\ \mathbf{\Gamma}_1 &= \sigma_1, \quad \mathbf{\Gamma}_2 = \sigma_2 \quad \mathbf{\Gamma}_3 = i\sigma_3. \end{aligned} \quad (3.19)$$

These generators belong to the $u(1)$ and $su(1, 1)$ algebras ($\mathbf{\Gamma}_\mu^+ = -\sigma_3 \mathbf{\Gamma}_\mu \sigma_3$) and define the corresponding group elements through the exponentials. These group elements can be obtained from the matrices C_L and C_R^1 (see Eqs. (3.9), (3.10)) using the substitution (3.18) and the fact that $\mathbf{\Gamma}_\mu$ satisfy the same multiplication relations as the ones given in Eq. (3.7). Thus, one can omit the charging symmetry subgroup $SO(2) \times SO(2, 1) \sim U(1) \times SU(1, 1)$ realized on the matrices C_L and C_R^1 and forming a total group of charging symmetries of Einstein-Maxwell theory if one starts in generation from the charging symmetry complete solutions of the stationary Einstein-Maxwell theory. The only resultative symmetry transformations are given by the matrix \mathcal{C}_R^2 , i.e. the string theory extension of the stationary Einstein-Maxwell theory is three-parametric.

Let us now calculate this extension. Let us put $\lambda_\mu^2 = \lambda_\mu$ for simplicity and apply Eqs. (2.3), (3.5), (3.9) and (3.11). The result reads:

$$\mathcal{Z}_1 = (z_1' \cosh \lambda + z_1'' \tilde{\lambda}_3) + (z_2' \tilde{\lambda}_1 - z_2'' \tilde{\lambda}_2) \sigma_1 + (z_2'' \tilde{\lambda}_1 + z_2' \tilde{\lambda}_2) \sigma_3 - (z_1'' \cosh \lambda - z_1' \tilde{\lambda}_3) \epsilon, \quad (3.20)$$

where $\tilde{\lambda}_\mu = \lambda^{-1} \sinh(\lambda) \lambda_\mu$, whereas \mathcal{Z}_2 can be obtained from \mathcal{Z}_1 using the substitution $z_1 \leftrightarrow z_2$. In Appendix B one can find a material related to calculation of the Ernst matrix potentials and a (partial) encoding of the σ -model information the language of the physical field components.

4 Generation using complex potentials

Let us consider the 2×2 nonconstrained complex matrix \mathbf{Z} ; it has the same number of degrees of freedom as the potential \mathcal{Z} . Let also \mathbf{C}_L is the $U(1)$ group factor and both the 2×2 matrices \mathbf{C}_R^1 and \mathbf{C}_R^2 parametrizes the group $SU(1, 1)$. Then the transformation

$$\mathbf{Z} \rightarrow \mathbf{C}_L \mathbf{C}_R^{2T} \mathbf{Z} \mathbf{C}_R^1 \quad (4.1)$$

realizes the $U(1) \times SU(1, 1) \times SU(1, 1)$ symmetry in action on the matrix \mathbf{Z} . We would like to use the group isomorphisms $U(1) \sim SO(2)$ and $SU(1, 1) \sim SO(2, 1)$ to parametrize the matrix \mathbf{Z} by the components of \mathcal{Z} in such a way when Eq. (4.1) will be a complex equivalent of Eq. (2.3). In the case of realization of this program we will have a formalism of the complex matrix potential which transforms linearly under the action of the charging symmetry subgroup of transformations. This new formalism will be the most compact and actually convenient for generation of asymptotically flat solutions of the theory. Moreover, it will be more natural than the real one in respect to string theory symmetry extension of the stationary Einstein-Maxwell theory which is naturally parametrized by the complex potential \mathbf{z} (see Eq. 3.13).

Our plan is the following: we will calculate the functional representation for the generators ‘1’ and ‘2’ in real (Γ_1^a and Γ_2^a in terms of the \mathcal{Z} -components) and complex ($\mathbf{\Gamma}_1^a$ and $\mathbf{\Gamma}_2^a$ in terms of the \mathbf{Z} -components) variables and will identify the corresponding generators using the proposing functional dependence between the real and complex variables. After that we will demand a proportionality $\Gamma_0 \sim \mathbf{\Gamma}_0$ for utilization of the $U(1)$ symmetry. The last step will be related with the convenient fixation of ‘gauge’ in the obtained solution - the explicit parametrization of \mathbf{Z} in terms of \mathcal{Z} components. Of course, we will not additionally consider the equality $\Gamma_3^a = \mathbf{\Gamma}_3^a$, because it will automatically take place in view of the previous steps and the commutation relations.

In order to realize this plan let us parametrize the potentials \mathbf{Z} and \mathcal{Z} as

$$\mathbf{Z} = \begin{pmatrix} z_1 & z_3 \\ z_4 & z_2 \end{pmatrix}, \quad \mathcal{Z} = \begin{pmatrix} \zeta_1 & \zeta_3 & \zeta_5 & \zeta_7 \\ \zeta_4 & \zeta_2 & \zeta_8 & \zeta_6 \end{pmatrix} \quad (4.2)$$

and calculate the generators in both the representations. From Eq. (2.3) it follows that the infinitesimal transformations of the potential \mathcal{Z} read:

$$\delta_\mu^a \mathcal{Z} = \mathcal{Z} \Gamma_\mu^a, \quad \delta_0 \mathcal{Z} = \Gamma_0 \mathcal{Z}, \quad (4.3)$$

so by the help of Eq. (4.2) one obtains for the real form of the functional generators the

following expressions:

$$\begin{aligned}
\Gamma_1^1 &= \zeta_5 \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial}{\partial \zeta_5} + \zeta_6 \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial}{\partial \zeta_6} + \zeta_3 \frac{\partial}{\partial \zeta_7} + \zeta_7 \frac{\partial}{\partial \zeta_3} + \zeta_4 \frac{\partial}{\partial \zeta_8} + \zeta_8 \frac{\partial}{\partial \zeta_4}, \\
\Gamma_2^1 &= -\zeta_7 \frac{\partial}{\partial \zeta_1} - \zeta_1 \frac{\partial}{\partial \zeta_7} + \zeta_2 \frac{\partial}{\partial \zeta_8} + \zeta_8 \frac{\partial}{\partial \zeta_2} - \zeta_4 \frac{\partial}{\partial \zeta_6} - \zeta_6 \frac{\partial}{\partial \zeta_4} + \zeta_5 \frac{\partial}{\partial \zeta_3} + \zeta_3 \frac{\partial}{\partial \zeta_5}; \\
\Gamma_1^2 &= \zeta_7 \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial}{\partial \zeta_7} + \zeta_8 \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial}{\partial \zeta_8} + \zeta_5 \frac{\partial}{\partial \zeta_3} + \zeta_3 \frac{\partial}{\partial \zeta_5} + \zeta_6 \frac{\partial}{\partial \zeta_4} + \zeta_4 \frac{\partial}{\partial \zeta_6}, \\
\Gamma_2^2 &= \zeta_5 \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial}{\partial \zeta_5} - \zeta_7 \frac{\partial}{\partial \zeta_3} - \zeta_3 \frac{\partial}{\partial \zeta_7} + \zeta_8 \frac{\partial}{\partial \zeta_4} + \zeta_4 \frac{\partial}{\partial \zeta_8} - \zeta_6 \frac{\partial}{\partial \zeta_2} - \zeta_2 \frac{\partial}{\partial \zeta_6}; \\
\Gamma_0 &= \zeta_1 \frac{\partial}{\partial \zeta_4} - \zeta_4 \frac{\partial}{\partial \zeta_1} + \zeta_3 \frac{\partial}{\partial \zeta_2} - \zeta_2 \frac{\partial}{\partial \zeta_3} + \zeta_5 \frac{\partial}{\partial \zeta_8} - \zeta_8 \frac{\partial}{\partial \zeta_5} + \zeta_7 \frac{\partial}{\partial \zeta_6} - \zeta_6 \frac{\partial}{\partial \zeta_7}. \tag{4.4}
\end{aligned}$$

Then, from Eq. (4.1) for the infinitesimal transformations of the potential \mathbf{Z} one obtains:

$$\delta_\mu^1 \mathbf{Z} = \mathbf{Z} \Gamma_\mu^1, \quad \delta_\mu^2 \mathbf{Z} = \Gamma_\mu^{2T} \mathbf{Z}, \quad \delta_0 \mathbf{Z} = \Gamma_0 \mathbf{Z}, \tag{4.5}$$

and, in view of Eq. (4.2), the corresponding complex generators read:

$$\begin{aligned}
\Gamma_1^1 &= z_3 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_4}, \\
\Gamma_2^1 &= i \left(z_3 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_3} + z_2 \frac{\partial}{\partial z_4} - z_4 \frac{\partial}{\partial z_2} \right); \\
\Gamma_1^2 &= z_4 \frac{\partial}{\partial z_1} + z_1 \frac{\partial}{\partial z_4} + z_3 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_3}, \\
\Gamma_2^2 &= i \left(z_4 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_4} + z_2 \frac{\partial}{\partial z_3} - z_3 \frac{\partial}{\partial z_2} \right); \\
\Gamma_0 &= i \left(z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_4} \right), \tag{4.6}
\end{aligned}$$

where only the holomorphic part is presented. For our purposes it will be convenient to have a linear \mathbf{Z} - \mathcal{Z} dependence. In fact it is possible to search it using an ansatz $z_{i'} = \mathcal{D}_{i'i''} \zeta_{i''}$, where $i' = 1, \dots, 4$ and $i'' = 1, \dots, 8$. The process of identification of the corresponding generators is trivial, it leads to establishing of different relations between the matrix \mathcal{D} components. The resulting parametrization of \mathbf{Z} in terms of \mathcal{Z} components can be represented in the following form: let us introduce the quantities

$$\hat{z}_1 = \zeta_1 + i\zeta_3 - \sigma(\zeta_2 - i\zeta_4),$$

$$\begin{aligned}
\hat{z}_2 &= \zeta_1 - i\zeta_3 + \sigma(\zeta_2 + i\zeta_4), \\
\hat{z}_3 &= \zeta_5 + i\zeta_7 - \sigma(\zeta_6 - i\zeta_8), \\
\hat{z}_4 &= \zeta_5 - i\zeta_7 + \sigma(\zeta_6 + i\zeta_8),
\end{aligned} \tag{4.7}$$

where the real constant σ is the coefficient in the equality $\Gamma_0 = \sigma\mathbf{\Gamma}_0$. Then all the generators become identified if $\sigma = \pm 1$ and

$$\mathbf{Z} = \begin{pmatrix} \mathcal{D}_{11}\hat{z}_1 + \mathcal{D}_{15}\hat{z}_4 & \mathcal{D}_{11}\hat{z}_3 + \mathcal{D}_{15}\hat{z}_2 \\ \mathcal{D}_{21}\hat{z}_4 + \mathcal{D}_{25}\hat{z}_1 & \mathcal{D}_{21}\hat{z}_2 + \mathcal{D}_{25}\hat{z}_3 \end{pmatrix}. \tag{4.8}$$

Here the parameters $\mathcal{D}_{11}, \mathcal{D}_{15}, \mathcal{D}_{21}, \mathcal{D}_{25}$ are restricted only by the demanding that the functional dependence of $z_{i'}, \bar{z}_{i''}$ on $\zeta_{i''}$ must be nondegenerated. In fact the freedom in choice of these parameters can be considered as some gauge freedom. Actually, let us consider the transformation

$$\mathbf{Z} \rightarrow \mathbf{L}\mathbf{Z}\mathbf{R}, \tag{4.9}$$

with the nondegenerated constant matrices \mathbf{L} and \mathbf{R} . It induces the following transformations of the group matrices of the following form:

$$\mathbf{C}_R^1 \rightarrow \mathbf{R}\mathbf{C}_R^1\mathbf{R}^{-1}, \quad \mathbf{C}_R^2 \rightarrow \mathbf{L}^T\mathbf{C}_R^2\mathbf{L}^{-1,T}. \tag{4.10}$$

Such transformations do not change the underlying generator multiplication relations and leave our scheme of generator identifications unchanged. It is easy to prove that it is possible to take the matrices \mathbf{L} and \mathbf{R} in such a way that finally $z_{i'} = \hat{z}_{i'}/2$. Also it is convenient to take $\sigma = -1$. Then

$$\mathbf{Z} = \frac{1}{2} \begin{pmatrix} \zeta_1 + \zeta_2 + i(\zeta_3 - \zeta_4) & \zeta_5 + \zeta_6 + i(\zeta_7 - \zeta_8) \\ \zeta_5 - \zeta_6 - i(\zeta_7 + \zeta_8) & \zeta_1 - \zeta_2 - i(\zeta_3 + \zeta_4) \end{pmatrix}. \tag{4.11}$$

Now let us consider the generation starting from the stationary Einstein-Maxwell fields. In this case (see Eqs. (3.11), (4.2))

$$\mathbf{Z}_{EM} = \begin{pmatrix} z \\ 0 \end{pmatrix}, \tag{4.12}$$

where the 1×2 matrix z is the Einstein-Maxwell potential from Eq. (3.14). Taking into account the reasons given in the previous section and omit \mathbf{C}_L and \mathbf{C}_R^1 transformations and also putting $\mathbf{C}_R^2 = \mathbf{C}$, one obtains that

$$\mathbf{Z} = \mathbf{C}^T\mathbf{Z}_{EM}, \tag{4.13}$$

where

$$\mathbf{C} = \cosh \lambda + \frac{\sinh \lambda}{\lambda} \hat{\lambda}, \quad (4.14)$$

and $\hat{\lambda} = \lambda_\mu \mathbf{\Gamma}_\mu$. Equations (4.11)-(4.14) completely realize the scheme of generation of the string theory solutions starting from the stationary Einstein-Maxwell fields in framework of the complex matrix potential formalism.

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5 Conclusion

In this article it is developed a formalism for generation of the stationary solutions of four-dimensional low-energy heterotic string theory with two Maxwell fields starting from the stationary Einstein-Maxwell theory. This formalism is the most general in respect to the stationary charging symmetry subgroup; it realizes the $SU(1, 1)$ -covariant string theory extension of the charging symmetry complete Einstein-Maxwell stationary solution space. Both the 2×4 real and 2×2 complex matrix potential based approaches are developed; a relation between them is given by Eq. (4.11). In the forthcoming publications I hope to generalize some results to the general case of arbitrary value of d and n of the heterotic string gravity model (2.1), and also to give some explicit examples of generation of the concrete solutions. Here it is useful only to stress that the new generation technique immediately gives the action of charging symmetries on the charges of asymptotically flat solutions. Actually, it is easy to prove that the potential

$$z = \frac{q}{R} \quad (5.1)$$

where q is the complex constant 1×2 matrix, together with the 3-dimensional line element

$$ds_3^2 = dR^2 + (R^2 - I) d\Omega^2, \quad (5.2)$$

where $I = qq^+$, is the solution of motion equations for the system (3.13). This asymptotically flat solution is in fact the Reissner-Nordstrom one (see [2]), and the matrix q components can be easily related to the mass, parameter NUT and also to electric and magnetic charges

of this point-like source. It is easy to see that the transformations \mathbf{C}_L and \mathbf{C}_R^1 provide the pure q -reparametrizations and can actually be omitted without loss of any generality. The remaining ‘second’ $SU(1, 1)$ transformation (we use the language of complex representation for definiteness) acts as

$$\mathbf{Q} \rightarrow \mathbf{C}^T \mathbf{Q} \quad (5.3)$$

on the charge matrix of the solution, defined accordingly the relation

$$\mathbf{Z} = \frac{\mathbf{Q}}{R}. \quad (5.4)$$

In this case Eq. (5.4) gives exact value of the \mathbf{Z} -potential. In the general case of asymptotically flat solution Eq. (5.4) defines the monopole term which is nonzero for the charged solutions. In this general situation Eq. (5.3) preserves its previous exact sense of the heterotic string theory generalization of the effective Einstein-Maxwell theory system of charges.

At the end of our discussion let us note that using the developed generation technique one obtains only some subspace of the stationary solution space of the theory under consideration. Actually, in general situation the matrix potential \mathbf{Z} is a complex matrix field free of any nondynamical restrictions. However, for \mathbf{Z} generated from \mathbf{Z}_{EM} one obtains that $\det \mathbf{Z} = 0$ identically, as it immediately follows from Eq. (4.13). The same conclusion can be done about the generality of charge characteristics of the generated solution. This opportunity seems interesting in the context of concrete work when often the real number of independent parameters of the constructed solution remains hidden in complicated notations. Our formalism guarantees clear form of all results in view of its explicit charging symmetry covariance.

Appendix A

The components of the original multidimensional fields can be embedded into three groups. The first group consists of the 3-dimensional scalars; these are the matrices G , B and A with the components G_{mk} , B_{mk} and $A_{mI} = A_m^I$. Their dimensions are $d \times d$, $d \times d$ and $d \times n$ respectively. Also there is a scalar field

$$\phi = \Phi - \ln \sqrt{-\det G}. \quad (\text{A.1})$$

The second group contains 3-vectors columns \vec{A}_1 , \vec{A}_2 and \vec{A}_3 of the dimension $d \times 1$, $d \times 1$ and $n \times 1$ respectively. Their components read:

$$(\vec{A}_1)_{m\mu} = (G^{-1})_{mk} G_{k,d+\mu},$$

$$\begin{aligned}
(\vec{A}_2)_{m\mu} &= B_{m,d+\mu} - B_{mn} (\vec{A}_1)_{n\mu} + \frac{1}{2} A_{mI} (\vec{A}_3)_{I\mu}, \\
(\vec{A}_3)_{I\mu} &= -A_{d+\mu}^I + A_{mI} (\vec{A}_1)_{m\mu}.
\end{aligned} \tag{A.2}$$

The third group is the group of 3-dimensional tensor fields; there are two ones:

$$\begin{aligned}
h_{\mu\nu} &= e^{-2\phi} \left[G_{\mu\nu} - G_{mk} (\vec{A}_1)_{m\mu} (\vec{A}_1)_{k\nu} \right], \\
b_{\mu\nu} &= B_{\mu\nu} - B_{mk} (\vec{A}_1)_{m\mu} (\vec{A}_1)_{k\nu} - \frac{1}{2} \left[(\vec{A}_1)_{m\mu} (\vec{A}_2)_{m\nu} - (\vec{A}_1)_{m\nu} (\vec{A}_2)_{m\mu} \right].
\end{aligned} \tag{A.3}$$

Following [5] we put $b_{\mu\nu} = 0$. Then, using the motion equations it is possible to ‘dualize’ the vector fields; let u , v and s are the $d \times 1$, $d \times 1$ and $d \times n$ (pseudo)scalar 3-fields related to the corresponding vector ones through the relations

$$\begin{aligned}
\nabla \times \vec{A}_1 &= e^{2\phi} G^{-1} \left[\nabla u + \left(B + \frac{1}{2} A A^T \right) \nabla v + A \nabla s \right], \\
\nabla \times \vec{A}_2 &= e^{2\phi} G \nabla v - \left(B + \frac{1}{2} A A^T \right) \nabla \times \vec{A}_1 + A \nabla \times \vec{A}_3, \\
\nabla \times \vec{A}_3 &= e^{2\phi} (\nabla s + A^T \nabla v) + A^T \nabla \times \vec{A}_1.
\end{aligned} \tag{A.4}$$

Finally, the set of the effective 3-dimensional scalars and pseudoscalars contains the quantities G , B , A , ϕ , u , v and s . They parametrize the heterotic string theory induced nonlinear σ -model coupled to gravity.

The Ernst matrix potential formulation is based on the use of two matrices

$$\mathcal{X} = \begin{pmatrix} -e^{-2\phi} + v^T X v + v^T A s + 1/2 s^T s & v^T X - u^T \\ X v + u + A s & X \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix}, \tag{A.5}$$

where $X = G + B + 1/2 A A^T$ ([7],[8]); in terms of them the effective 3-dimensional action reads:

$$S_3 = \int dx \sqrt{h} \left\{ -R_3 + \text{Tr} \left[\frac{1}{4} (\nabla \mathcal{X} - \nabla \mathcal{A} \mathcal{A}^T) \mathcal{G}^{-1} (\nabla \mathcal{X}^T - \mathcal{A} \nabla \mathcal{A}^T) \mathcal{G}^{-1} + \frac{1}{2} \nabla \mathcal{A}^T \mathcal{G}^{-1} \nabla \mathcal{A} \right] \right\}, \tag{A.6}$$

where

$$\mathcal{G} = 1/2 (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T). \tag{A.7}$$

The null-curvature matrix formulation is related to the matrix \mathcal{M} , which can be defined using the Ernst matrix potentials [11]:

$$\mathcal{M} = \begin{pmatrix} \mathcal{G}^{-1} & \mathcal{G}^{-1}\mathcal{X} - 1 & \mathcal{G}^{-1}\mathcal{A} \\ \mathcal{X}^T\mathcal{G}^{-1} - 1 & \mathcal{X}^T\mathcal{G}^{-1}\mathcal{X} & \mathcal{X}^T\mathcal{G}^{-1}\mathcal{A} \\ \mathcal{A}^T\mathcal{G}^{-1} & \mathcal{A}^T\mathcal{G}^{-1}\mathcal{X} & \mathcal{A}^T\mathcal{G}^{-1}\mathcal{A} + 1 \end{pmatrix}. \quad (\text{A.8})$$

This matrix parametrizes the coset $O(d+1, d+1+n)/O(d+1) \times O(d+1+n)$ in view of the number of its independent components and the identical satisfaction of the relations

$$\mathcal{M}^T = \mathcal{M}, \quad \mathcal{M}\mathcal{L}\mathcal{M} = \mathcal{L}, \quad (\text{A.9})$$

where

$$\mathcal{L} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (\text{A.10})$$

In terms of the curvature matrix the 3-action takes the following form:

$$S_3 = \int dx \sqrt{h} \left\{ -R_3 + \frac{1}{8} \text{Tr} \left[\left(\nabla \mathcal{M} \mathcal{M}^{-1} \right)^2 \right] \right\}. \quad (\text{A.11})$$

The \mathcal{Z} -potential formulation can also be defined through the Ernst matrix potentials. To do it, let us define the matrices

$$\mathcal{Z}_1 = 2(\mathcal{X} + \Sigma)^{-1} - \Sigma, \quad \mathcal{Z}_2 = \sqrt{2}(\mathcal{X} + \Sigma)^{-1} \mathcal{A}. \quad (\text{A.12})$$

Then

$$\mathcal{Z} = (\mathcal{Z}_1 \ \mathcal{Z}_2). \quad (\text{A.13})$$

The \mathcal{Z} -expressed effective 3-dimensional action is given by Eq. (2.2).

Appendix B

It is easy to verify that the matrices

$$g_0 = 1, \quad g_1 = \sigma_1, \quad g_2 = \sigma_3, \quad g_3 = -\epsilon \quad (\text{B.1})$$

satisfy the same multiplicative relations as the in Eq. (3.7). In fact the set $\{g_{\aleph}\}$ ($\aleph = 0, \dots, 3$) give the generators of the $sl(2, R)$ algebra which is isomorphic to $so(1, 2)$. The multiplication table can be used for the explicit computation of the Ernst matrix potentials and for the following obtaining of components of the physical fields. In this computation it will be convenient to use the decompositions of all the 2×2 matrices in respect to the basis (B.1) (for example, the decomposition $\mathcal{Z}^a = \mathcal{Z}_{\aleph}^a g_{\aleph}$ is given in Eq. (3.20)).

In computation of the Ernst matrix potentials one must take into account their 2×2 matrix dimensionality; from this fact and Eq. (A.12) it follows that

$$\mathcal{X} = -1 + \frac{2}{\Delta}(-\mathcal{Z}_1 + \det \mathcal{Z}_1), \quad \mathcal{A} = -\frac{\sqrt{2}}{\Delta}(1 - \mathcal{Z}_1^*) \mathcal{Z}_2, \quad (\text{B.2})$$

where $\Delta = 1 - \text{Tr} \mathcal{Z}_1 + \det \mathcal{Z}_1$ and $\mathcal{Z}_1^* = -\epsilon \mathcal{Z}_1^T \epsilon = \mathcal{Z}_{1,0} g_0 - \mathcal{Z}_{1,\mu} g_{1,\mu}$. Here $\mathcal{Z}_1^* \mathcal{Z}_2 = (\mathcal{Z}_1^* \mathcal{Z}_2)_{\aleph} g_{\aleph}$, where

$$\begin{aligned} (\mathcal{Z}_1^* \mathcal{Z}_2)_{,0} &= \mathcal{Z}_{1,0} \mathcal{Z}_{2,0} - \mathcal{Z}_{1,1} \mathcal{Z}_{2,1} - \mathcal{Z}_{1,2} \mathcal{Z}_{2,2} + \mathcal{Z}_{1,3} \mathcal{Z}_{2,3}, \\ (\mathcal{Z}_1^* \mathcal{Z}_2)_{,1} &= \mathcal{Z}_{1,0} \mathcal{Z}_{2,1} - \mathcal{Z}_{2,0} \mathcal{Z}_{1,1} + \mathcal{Z}_{1,2} \mathcal{Z}_{2,3} - \mathcal{Z}_{1,3} \mathcal{Z}_{2,2}, \\ (\mathcal{Z}_1^* \mathcal{Z}_2)_{,2} &= \mathcal{Z}_{1,0} \mathcal{Z}_{2,2} - \mathcal{Z}_{2,0} \mathcal{Z}_{1,2} + \mathcal{Z}_{1,3} \mathcal{Z}_{2,1} - \mathcal{Z}_{1,1} \mathcal{Z}_{2,3}, \\ (\mathcal{Z}_1^* \mathcal{Z}_2)_{,3} &= \mathcal{Z}_{1,0} \mathcal{Z}_{2,3} - \mathcal{Z}_{2,0} \mathcal{Z}_{1,3} - \mathcal{Z}_{1,1} \mathcal{Z}_{2,2} + \mathcal{Z}_{1,2} \mathcal{Z}_{2,1}. \end{aligned} \quad (\text{B.3})$$

Also it is easy to see that $\text{Tr} \mathcal{Z}_1 = 2\mathcal{Z}_{1,0}$ and $\det \mathcal{Z}_1 = \mathcal{Z}_{1,0}^2 - \mathcal{Z}_{1,1}^2 - \mathcal{Z}_{1,2}^2 + \mathcal{Z}_{1,3}^2$. Then, using definitions of the potentials \mathcal{X} and \mathcal{A} (see Eq. (A.5)), one can calculate all the scalar and pseudoscalar variables. For example, for the electric potentials one obtains:

$$\begin{aligned} A_t^1 &= -\frac{\sqrt{2}}{\Delta} [\mathcal{Z}_{2,1} + \mathcal{Z}_{2,3} - (\mathcal{Z}_1^* \mathcal{Z}_2)_{,1} - (\mathcal{Z}_1^* \mathcal{Z}_2)_{,3}], \\ A_t^2 &= -\frac{\sqrt{2}}{\Delta} [\mathcal{Z}_{2,0} - \mathcal{Z}_{2,2} - (\mathcal{Z}_1^* \mathcal{Z}_2)_{,0} + (\mathcal{Z}_1^* \mathcal{Z}_2)_{,2}]. \end{aligned} \quad (\text{B.4})$$

Then, calculating \mathcal{G} accordingly Eq. (A.7) by exploring the multiplication table (3.7), one obtains the remaining nondualizing components. Here we will not give them here in view of their complicated form as well as the expressions for the redualized 3-vectors and the related physical field components. They will be presented in the following publications related to the concrete generations of new solutions of low-energy heterotic string theory.

References

- [1] E. Kiritsis, “Introduction to superstring theory”, Leuven Univ. Pr. (1998).

- [2] D. Kramer, H. Stephani, M. MacCallum, E. Herlt, “Exact solutions of the Einstein field equations”, Deutscher Verlag der Wissenschaften, Berlin, (1980).
- [3] J. Maharana, J.H. Schwarz, Nucl. Phys. **B390** (1993) 3.
- [4] A. Sen, Int. J. Mod. Phys. **A9** (1994) 3707.
- [5] A. Sen, Nucl. Phys. **B434** (1995) 179.
- [6] A.K. Biswas, A. Kumar, K. Ray, Nucl. Phys. **B453** (1995) 181.
- [7] A. Herrera-Aguilar, O.V. Kechkin, Int. J. Mod. Phys. **A13** (1998) 393.
- [8] A. Herrera-Aguilar, O.V. Kechkin, Phys. Rev. **D59** (1999) 124006.
- [9] D. Youm, Phys. Rept. **316** (1999) 1.
- [10] O.V. Kechkin, Mod. Phys. Lett. **A14** (1999) 1599; Gen. Rel Grav. **31** (1999) 1087; O.V. Kechkin, M.V. Yurova, Gen. Rel Grav. **29** (1997) 1283; D.V. Galtsov, O.V. Kechkin, Phys. Rev. **D59** (1994) 7399.
- [11] O.V. Kechkin, “New matrix formalism for heterotic string theory on a torus” gr-qc/0012090.
- [12] P. Breitenlohner, D. Maison, G.W. Gibbons, Commun. Math. Phys. **120** (1988) 295.
- [13] M. Cvetič, C.M. Hull, Nucl. Phys. **B480** (1996) 296.
- [14] R. Kallosh, D. Kastor, T. Ortin, T. Torma, Phys. Rev. **D50** (1994) 6384.
- [15] P.O. Mazur, Acta Phys. Pol. **B14** (1983) 219.
- [16] F. J. Ernst, Phys. Rev. **167** 5 (1968) 1175.